Maximally-flat Filters in Waveguide

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Microwave radio relay repeaters require the use of band-pass filters which match closely the impedances of the interconnecting transmission lines and which suppress adjacent channels adequately. A type of structure called a *Maximally-Flat* filter meets these requirements.

The ladder network which gives a maximally-flat insertion loss characteristic is discussed and several methods of achieving its counterpart in microwave transmission lines are presented. Resonant cavities are used to simulate tuned circuits and the necessary formulas relative to this approximate equivalence are given.

Experimental data confirm the theory and show that this technique yields remarkable impedance matches.

INTRODUCTION

WE USUALLY associated tbe word *filter* with any device which is selective. The electric wave filter has that property which enables it to transmit energy in one band or bands of frequencies and to inhibit energy in other bands. Selectivity is the result of either selective absorption[1,2*] or selective reflection. This paper discusses a special case of the classical lossless transducer which derives its selective properties entirely from selective reflection. The insertion loss of this type of filter can be analyzed in terms of the input reflection coefficient and the input standing wave ratio.

In many applications of lossless filters it is desirable to obtain a characteristic such that the insertion loss, and hence the reflection coefficient, is small over as wide a band as possible. A special case described here, referred to as a maximally-flat filter, has a loss characteristic such that a maximum number of its derivatives are zero at midband. While the maximally-flat type of characteristic does not give the smallest possible reflections over a finite pass band, it does give small reflections, and has added advantages of simplicity in design and in many cases less transient distortion than filters giving smaller reflections.

The desirable characteristics of maximally-flat filters have long been realized.[1,3] Mr. W. R. Bennett[4] of these Laboratories derived the constants for a maximally-flat ladder network in the late 20's, and gave simple expressions for the element values. Butterworth,[5] Landon[6] and Wallman[7] have treated maximally-flat filter-amplifiers in which the filter sections are separated by amplifier tubes. Darlington[8] bas considered tbe general case of four terminal filters wbich have insertion loss characteristics that can be prescribed, but he places the empbasis more on filters that have tolerable

* A list of selected references appears at the end of the paper.

ripples in the pass-band than on maximally-flat structures. The work of Bennett will be followed closely not only because it came first, but also because it is easy to understand.

Bennett expresses the values of the filter branches in terms of their cutoff frequencies, which in turn bear a relationship to the cutoff frequencies of the total filter. In the language of one who is familiar with microwave technique,[9, 10, 11, 12] the values of the filter branches can be expressed in terms of the loaded Q's of the cavities, which in turn bear a relationship to the loaded Q of the total filter. A simple mathematical expression connects the loaded Q to the cutoff wavelengths.

At low frequencies the band-pass maximally-flat filter is made up of resonant branches connected alternately in series and in parallel. The microwave analogue of this configuration is obtained by the use of shunt resonant cavities that are spaced approximately a quarter wavelength apart in the waveguide. Use is made of the impedance inverting property of a quarter wave line, thereby eliminating the necessity of using both series and parallel branches.

The resonant cavity in the waveguide resembles a shunt resonant tuned circuit,[13] but is different in several minor respects. An analysis of these differences reveals the corrective measures that are necessary in order that the simulation shall be sufficiently accurately attained.

The first part of the paper deals with the concepts of loaded Q and resonant filter branches of both the series and the parallel types. Admittance and impedance functions, as well as expressions for the insertion loss, are given using these terms, and the relationship between loaded Q and cutoff frequencies is stated. This concept of loaded Q is then introduced to describe the performance of a complete maximally-flat filter in terms of its cutoff frequencies. The insertion loss is then given as a simple expression containing the total Q and the resonant frequency. The Q's of all the branches are derived from the total Q in simple terms. The connection between the insertion loss and the input standing wave ratio is then discussed before turning to the actual design problem.

Next the paper deals with the application of the filter theory to waveguide technique. The limitations of the quarter-wave coupling lines are pointed out and the added selectivity due to them is derived.

Then the paper compares microwave resonant cavities with parallel-tuned circuits. Formulas are given which relate the geometrical configuration to the loaded Q, the resonant frequency and the excess phase of the cavities. Three types of cavities are treated: those using inductive posts, inductive irises and capacitive irises.

Finally, the measured results on a four-cavity maximally-flat filter in $1'' \times 2''$ waveguide are presented and compared with the original design points.

As a further confirmation of the theory, the experimental results on another and longer waveguide filter consisting of fifteen resonant cavities and fourteen connecting lines are given. The conclusion is reached that maximally-flat waveguide filters can be designed to have excellent impedance match and off-band suppression qualities.

Notation

a	Width of waveguide.
b	Height of waveguide.
B	Normalized susceptance.
c	Velocity of light in free space.
C_r	Capacitance in the r^{th} branch of a filter.
d	Width of iris opening.
d	Diameter of post in waveguide.
δ	A small number $\ll 1$.
ϵ	Base of natural logarithms.
f	Frequency.
f_0	Resonant frequency.
f_c	Frequency at half power point.
f_{cw}	Cutoff frequency of waveguide.
G	Terminating conductance of filter.
θ	$\dfrac{2\pi \ell}{\lambda}$
K	Susceptance parameter.
ℓ	Length of transmission line.
ℓ'	Length of line corresponding to excess phase of cavity.
ℓ_c	Length of line connecting two cavities.
λ_0	Resonant wavelength.
λ_c	Wavelength at half power point.
λ_g	Wavelength in transmission line.
λ_a	Wavelength in free space.
L_r	Inductance in r^{th} branch of a filter.
m	An integer, including zero.
n	Number of branches in filter.
P_0	Available power.
P_L	Power delivered to load.
Q	Loaded Q. The selectivity of a loaded circuit.
Q_r	Loaded Q of the r^{th} branch.
Q_T	Loaded Q of the total filter.
R	Terminating resistance of the filter.
s	Distance from center of waveguide.
S	Voltage standing wave ratio.

τ	Thickness of iris.
V_{max}	Maximum voltage on transmission line.
V_{min}	Minimum voltage on transmission line.
ω	Angular frequency.
Ω	Frequency parameter.
Y	Admittance.
Y_0	Surge admittance of transmission line.
Z	Impedance.
Z_0	Surge impedance of transmission line.

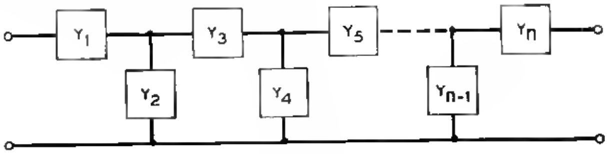


Fig. 1—Block diagram of a filter consisting of a ladder network.

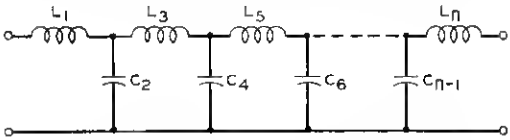


Fig. 2—Schematic diagram of a low-pass filter.

General

The art of designing filters which utilize lumped elements is well known. Desirable characteristics may be obtained by means of a ladder network of generalized admittances, such as is illustrated in Fig. 1. In particular a low-pass filter takes the configuration shown in Fig. 2, and a band-pass filter that of Fig. 3. In either case, certain frequency selectivity characteristics can be obtained when the individual branches are assigned definite values. The individual branches, L_1C_1, L_2C_2, in the bandpass filter consist of an inductance, L_r, and capacity, C_r, in series or shunt. For the specific case to be discussed in this paper, namely a filter consisting of lossless elements intended for insertion between a source having an internal resistance R and a receiver having the same resistance, analysis is simplified if a branch is described in terms of its resonant frequency and its loaded Q.* The

* The loaded Q of a resonant branch in such a filter is the reciprocal of its percentage band width measured to the half power points when that branch alone is fed by the same generator and has the same load resistance as that of the total filter.

resonant frequency of a branch is independent of the terminal resistance and is given by the relation

$$f_0 = \frac{1}{2\pi\sqrt{L_r C_r}} \tag{1}$$

The loaded Q of the branch, to be designated Q_r, is a function not only of the inductance and capacitance in the branch, but also of the resistance, R, of the terminations on the filter. For the series resonant branches, it is given by

$$Q_r = \frac{1}{2R}\sqrt{\frac{L_r}{C_r}} = \frac{\omega_0 L_r}{2R} \tag{2}$$

and for the parallel resonant hranches

$$Q_r = \frac{R}{2}\sqrt{\frac{C_r}{L_r}} = \frac{\omega_0 C_r}{2} R. \tag{3}$$

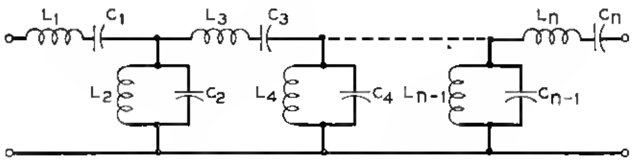


Fig. 3—Schematic diagram of a band-pass filter.

It may be noted that the loaded Q can be defined in terms of the insertion loss imposed by connecting the branch between a source and receiver each of resistance R. Analysis of such a circuit shows that

$$\frac{P_0}{P_L} = 1 + Q_r^2\left(\frac{f}{f_0} - \frac{f_0}{f}\right)^2 \tag{4}$$

where f is the frequency;

P_0 is the power available from a generator which has an internal resistance R;

P_L is the power delivered through the inserted branch to a load of resistance R.

At the cutoff frequency, f_c, defined as the frequency at which the power delivered to the load is half the available power, $\frac{P_0}{P_L} = 2$, whence

$$Q_r = \left|\frac{1}{\dfrac{f_c}{f_0} - \dfrac{f_0}{f_c}}\right| = \left|\frac{f_0}{f_{c_2} - f_{c_1}}\right|. \tag{5}$$

Written in terms of the wavelengths this becomes

$$Q_r = \frac{\dfrac{1}{\lambda_0}}{\dfrac{1}{\lambda_{c_2}} - \dfrac{1}{\lambda_{c_1}}}. \tag{6}$$

This equation is a convenient one to use later in the discussion on resonant cavities.

The normalized admittance of a single-shunt branch terminated by a resistance R can be expressed in terms of its resonant frequency and its Q; thus

$$YR = 1 + j2Q_r\left(\frac{f}{f_0} - \frac{f_0}{f}\right). \tag{7}$$

Similarly, the normalized impedance of a single-series branch terminated by a resistance R can be written

$$\frac{Z}{R} = 1 + j2Q_r\left(\frac{f}{f_0} - \frac{f_0}{f}\right). \tag{8}$$

The use of the term loaded Q thus has the advantage that expressions for normalized admittance and normalized impedance of shunt and series resonant circuits respectively are identical, as are also the corresponding expressions for their insertion loss functions.

Loss functions of complete filters can likewise be expressed in terms of a loaded Q defined for the complete filter. For example, the loss function of the particular type of filter called a "Maximally-flat" filter is given[4]

$$\frac{P_0}{P_L} = 1 + \left[\frac{\dfrac{f}{f_0} - \dfrac{f_0}{f}}{\dfrac{f_c}{f_0} - \dfrac{f_0}{f_c}}\right]^{2n} \tag{9}$$

where n is the number of resonant branches in the filter, and f_c is the cutoff frequency of the filter (half power points).

In consequence of the concept of loaded Q of the total filter, the loss function can be expressed as

$$\frac{P_0}{P_L} = 1 + \left[Q_T\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right]^{2n} \tag{10}$$

where the total Q_T of the filter is

$$Q_T = \left|\frac{1}{\dfrac{f_c}{f_0} - \dfrac{f_0}{f_c}}\right|. \tag{11}$$

For convenience, the bracketed term of equation 9 may be called Ω, a frequency parameter, whence

$$\Omega = \left[\frac{\dfrac{f}{f_0} - \dfrac{f_0}{f}}{\dfrac{f_c}{f_0} - \dfrac{f_0}{f_c}}\right] = Q_T\left(\frac{f}{f_0} - \frac{f_0}{f}\right) \tag{12}$$

and the loss function becomes

$$\frac{P_0}{P_L} = 1 + (\Omega)^{2n}. \tag{13}$$

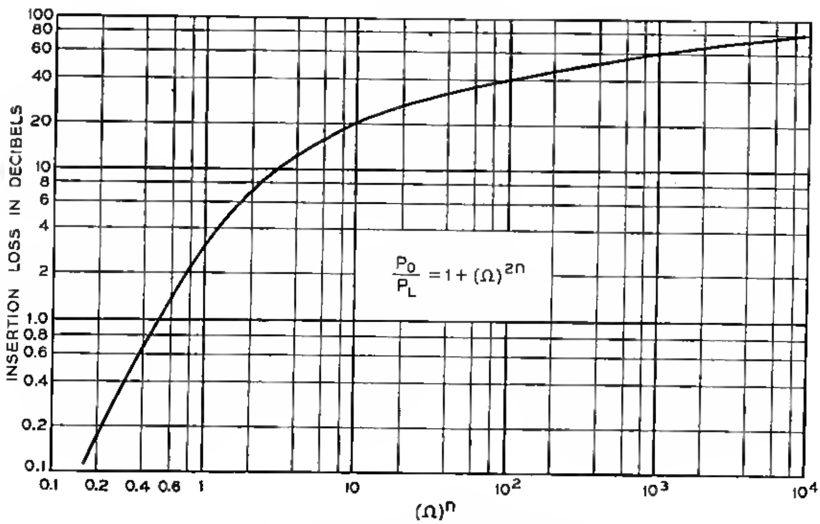


Fig. 4—Insertion loss of maximally-flat filters.

Maximally-Flat Filters

The loss function for maximally-flat filters as given in equation 13 is plotted in Fig. 4 where the insertion loss in db is used on the ordinate and Ω^n is used on the abscissa.

The ladder network which gives rise to this loss function consists of n resonant branches, as shown in Fig. 3, that are all tuned to the same frequency, but whose selectivities, or loaded Q's, are tapered from one end of the filter to the other according to the positive imaginary parts of the $2n$ roots of -1, according to the theories of Bennett[4] and Darlington.[8] These roots are expressed thus

$$\sin\left(\frac{2r-1}{2n}\right)\pi$$

where r is the number of the root, n is the total number of branches. Thus the selectivities of the branches follow the relation

$$Q_r = Q_T \sin\left(\frac{2r - 1}{2n}\right)\pi \tag{14}$$

where Q_T represents the selectivity of the total filter, and Q_r represents the required selectivity of the r^{th} branch, e.g., the selectivities of the first, second and third hranches are

$$\begin{aligned} Q_1 &= Q_T \sin\frac{\pi}{2n} \\ Q_2 &= Q_T \sin\frac{3\pi}{2n} \\ Q_3 &= Q_T \sin\frac{5\pi}{2n}. \end{aligned} \tag{15}$$

This type of filter is particularly practical when a filter is required to give more than a certain amount of insertion loss in an adjacent band, and less than another certain amount of insertion loss at the edges of the pass-band. Putting this information in equation 10 gives two equations containing two unknowns, Q_T, the selectivity of the total filter, and n, the number of branches needed to fulfill the stated requirements. The solution for n may be fractional, in which event the next higher integral value of n is chosen, and this value is used to determine the selectivity, Q_T, of the filter. From this, the selectivities of all the branches are determined in accordance with equation 14.

Standing Wave Ratio

An alternative way of specifying filter performance is to refer to the input impedance mismatch as a function of frequency. The impedance mismatch can be expressed in terms of the direct and the reflected waves and in terms of the standing wave ratio that exists along the transmission line that connects the properly terminated filter with its generator. The standing wave ratio and the insertion loss of a filter bear a definite relationship to each other if the filter is composed of purely reactive elements. This relationship is given by the formula

$$\frac{P_0}{P_L} = \frac{(S + 1)^2}{4S} \tag{16}$$

where S is the standing wave ratio, $\frac{V_{\max}}{V_{\min}}$, of the maximum voltage to the minimum voltage as measured along the transmission line.

When the filter characteristic is given by equation 13, the relationship between Ω, the frequency parameter, and the standing wave ratio can be expressed as

$$(\Omega)^n = \frac{S - 1}{2\sqrt{S}} . \tag{17}$$

This is shown graphically in Fig. 5, where the standing wave ratio is given in db $\left(20 \log_{10} \frac{V_{\max}}{V_{\min}}\right)$. This graph is used as an aid in the design of filters of

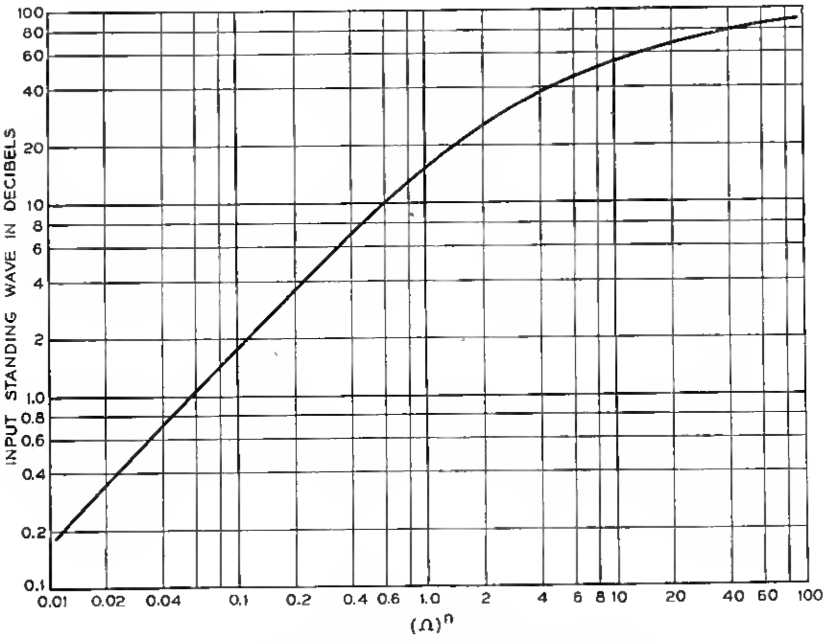


Fig. 5—Input standing wave ratio of maximally-flat filters.

this type, where the requirements are given in terms of the standing wave ratio. From this information the number of filter branches and the selectivity of the total filter can be determined, either from equation 17 or from Fig. 5.

Distributed Branches

It has been assumed that the mutual impedances of successive branches are all zero. At low frequencies this limitation may not be a serious one and the practical realization of the expected filter characteristics is accomplished by shielding properly one branch from another. However, as the

frequency is increased it becomes difficult to isolate the branches and undesirable mutual impedances arise which complicate the problem. In particular, in the microwave region, where waveguides are used, the physical size of each branch may be large compared with the wavelength and it is then impossible to lump all the branches at one place in the waveguide without encountering the complicated effect of mutual impedances.

A practical way of circumventing this difficulty is to distribute the branch circuits along the transmission line or waveguide at such distances that the mutual impedances become negligible. Then, however, the lengths of transmission line act as transducers, but since their properties are well understood and readily calculable this appears to be a practical solution. As a matter of fact, the impedance transforming properties of a length of transmission line can be used to advantage.[9, 13, 14] For instance, it is well known that a quarter wavelength of lossless line transforms a load impedance according to the relation

$$Z = \frac{Z_0^2}{Z_L} \tag{18}$$

where Z_0 is the surge impedance of the line and Z_L is the load impedance.

Hence if the load impedance consists of a series resonant circuit containing an inductance, a capacity and a resistance equal to Z_0 in series, the impedance at the input end of the quarter wavelength of line is given

$$Z = \frac{Z_0}{\left[1 + j2Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right]}. \tag{19}$$

The input admittance is

$$Y = Y_0\left[1 + j2Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right]. \tag{20}$$

As can be seen from equation 7, this is identical with the input admittance of a parallel tuned circuit whose terminating conductance is

$$G = Y_0. \tag{21}$$

The quarter-wave line likewise transforms a parallel circuit to a series circuit, as is illustrated in Fig. 6. This property of the quarter-wave line thus makes it possible to simulate a ladder network of alternate series and shunt branches by spacing shunt branches (or series branches) at quarter wavelength intervals along a transmission line, as illustrated in Fig. 7. The resonant frequencies and the selectivities of the branches are chosen as before.

Sometimes in practice a quarter wavelength may not be sufficient spacing to avoid mutual impedances arising between adjacent elements, in which event the connecting line may be increased to a higher odd multiple of quarter wavelength. This accentuates the frequency sensitivity of the connect-

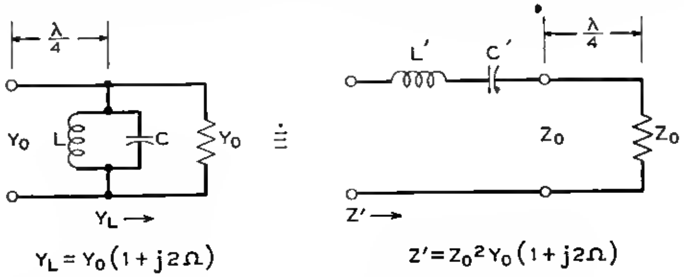


Fig. 6—Illustrating the impedance inverting property of a quarter wavelength of transmission line.

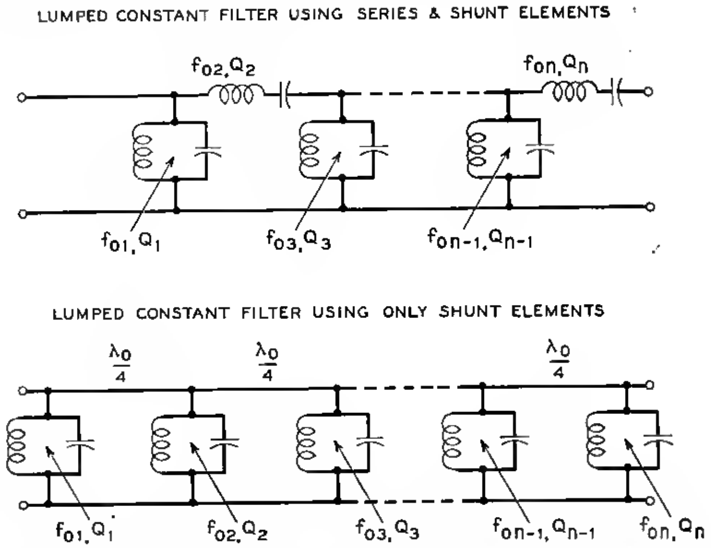


Fig. 7—Simulation of ladder network by shunt branches at quarter wave intervals.

ing line, but this effect can be taken into account by decreasing the selectivities of the branches themselves by appropriate amounts. In narrow-band filters this may be negligible, but in broad-band filters it may be considerable, as shown in the following analysis.

Selectivity of Connecting Lines

Consider a length of transmission line having a surge impedance $Z_0 = \frac{1}{Y_0}$ and terminated in a parallel resonant circuit containing an inductance, a

capacitance and a resistance equal to Z_0,* as in Fig. 6. The terminating admittance is given by the relation (See Eq. 7 and 9)

$$Y_L = Y_0(1 + j2\Omega). \tag{22}$$

The input admittance at the end of the length of line, ℓ, (nominally a quarter wavelength long) is given by the relation

$$\frac{Y}{Y_0} = \frac{(1 + j2\Omega) \cos \theta + j \sin \theta}{\cos \theta + j(1 + j2\Omega) \sin \theta} \tag{23}$$

where

$$\theta = \frac{2\pi\ell}{\lambda}$$

$$\ell = \text{length of line}$$

$$\lambda = \text{wavelength}$$

$$\Omega = Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right)$$

letting

$$\theta = \frac{\pi}{2}(1 + \delta) = \frac{\pi}{2} + \frac{\pi\delta}{2} \tag{24}$$

$$\cos \theta = -\sin \frac{\pi\delta}{2} \doteq -\frac{\pi\delta}{2} \tag{25}$$

$$\sin \theta = \cos \frac{\pi\delta}{2} \doteq 1 \tag{26}$$

where δ is a number small compared with 1. Then the admittance becomes

$$\frac{Y}{Y_0} \doteq j\frac{\pi\delta}{2} + \frac{1}{1 + j\left(2\Omega + \dfrac{\pi\delta}{2}\right)}. \tag{27}$$

This is the normalized input admittance of a circuit as shown in Fig. 8, where each end of an ideally inverting line is shunted by a tuned circuit whose normalized admittance is $j\dfrac{\pi\delta}{2}$.

From Eq. 24, setting $\dfrac{2\pi\ell}{\lambda_0} = \dfrac{\pi}{2}$, it follows that

$$\delta = \pm\left(\frac{f}{f_0} - 1\right) \doteq \left(\frac{f}{f_0} - \frac{f_0}{f}\right)(\tfrac{1}{2}). \tag{28}$$

* More generally, the terminating admittance can assume any value without affecting the result.

From equation 7, the admittance of the circuit is expressed in terms of its selectivity, thus

$$j\frac{\pi\delta}{2} = j2Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right). \tag{29}$$

Solving for the selectivity of this circuit, from Equations 28 and 29:

$$Q = \frac{\pi}{8}. \tag{30}$$

The selectivity of the coupling line can bence be counteracted by subtracting $\frac{\pi}{8}$ from the selectivities of the branches associated with it, provided

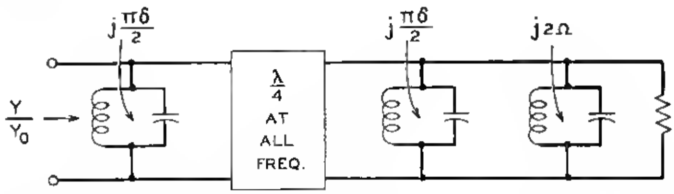


Fig. 8—Schematic diagram illustrating that the selectivity of a quarter wavelength of line can be represented by adding a tuned circuit to each end of an ideally inverting impedance transformer.

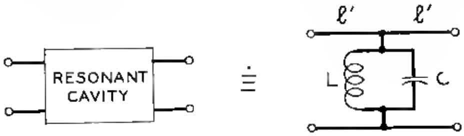
the coupling line is a quarter wavelengtb long. If it becomes necessary to use $\frac{3}{4}$ wavelength coupling lines, the selectivity of the line is tripled and $\frac{3\pi}{8}$ is subtracted from the selectivities of the associated branches.

Resonant Cavities

The foregoing analysis reviews the principles of the design of filters which use lumped-constant circuits distributed along a transmission line. These principles can be applied to the design of filters in waveguides, coaxial lines, or any other types of transmission lines, provided that these lines are sufficiently lossless, the band is sufficiently narrow and the branches themselves are realizable. In the microwave region the first two provisions are usually met witbout difficulty, as is also the third provision wben circuits witb distributed constants are used. It may be difficult to construct a coil and a condenser circuit for microwaves, but easy to construct a resonant cavity which displays some of the desirable properties of the tuned circuit. Resonant cavities are similar to lumped tuned circuits in two respects.[12, 13] They transmit a band of frequencies and they introduce a phase shift. An approximate equivalence is demonstrated in Appendix I, and is illustrated in Fig. 9, which depicts a resonant cavity as being nearly identical with a

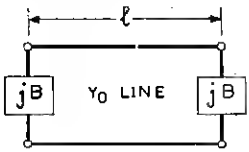
tuned circuit situated across the middle of a short length of transmission line. This short length of transmission line is added in order to account for an excess of phase shift associated with the resonant cavity, but it can readily be absorbed in the connecting line which otherwise would have been an odd quarter wavelength long.

The similarity between resonant cavities and resonant lumped circuits enables one to use the known art of designing lumped element filters to de-



$$\ell + 2\ell' = \frac{\lambda_g}{2}$$

Fig. 9—A resonant cavity is approximately equivalent to a resonant circuit shunted across a short length of transmission line.



$$\mathrm{TAN}\,\frac{2\pi\ell}{\lambda_0} = \frac{2}{B}$$

$$Q = \frac{\mathrm{ARC\ TAN}\,\frac{2}{B}}{2\,\mathrm{ARC\ SIN}\,\frac{2}{\sqrt{B^4+4B^2}}} \doteq \frac{\sqrt{B^4+4B^2}}{4}\,\mathrm{ARC\ TAN}\,\frac{2}{B}$$

Fig. 10—The resonant wavelength and the loaded Q of a cavity depend upon the normalized susceptance of the end obstacles and their separation.

sign filters which use resonant cavities, provided that the selectivity, the resonant frequency and the excess phase shift of the resonant cavity are known.

Resonant Wavelength and Loaded Q of Cavities

These properties can best be derived by considering one of the usual types of cavities, which consists of two obstacles or discontinuities separated by a length of transmission line. Such a cavity is shown schematically in Fig. 10. The obstacles at each end are assumed to be equal, and to have an unvarying susceptance BY_0, where Y_0 is the surge admittance of the con-

necting transmission line. This type of cavity is resonant when the relation is satisfied[9, 10]

$$\tan \frac{2\pi \ell}{\lambda_0} = \frac{2}{B} \tag{31}$$

where λ_0 is the resonant wavelength in the transmission line,
ℓ is the length of the cavity
B is the normalized susceptance of the end obstacles.

This resonance occurs at any number of wavelengths, but the 1st or 2nd longest wavelength at which resonance occurs is in the region which is usually of greatest interest.

The selectivity in this region is determined also by the value of the normalized susceptance, B, of the obstacles, and is given by the relation (See Appendix I)

$$Q = \frac{\arc\tan \frac{2}{B}}{2 \arc\sin \frac{2}{\sqrt{B^4 + 4B^2}}}. \tag{32}$$

This selectivity is based upon the wavelength, not the frequency parameter. In terms of the wavelength in the transmission line this is

$$Q = \left| \frac{\frac{2\pi\ell}{\lambda_{g0}}}{\frac{2\pi\ell}{\lambda_{gc1}} - \frac{2\pi\ell}{\lambda_{gc2}}} \right| \doteq \left| \frac{\lambda_{g0}}{\lambda_{gc1} - \lambda_{gc2}} \right| \tag{33}$$

where λ_{g0} is the wavelength of resonance in the transmission line and λ_{gc} is the wavelength at the half power points. If the phase velocity in the transmission line does not vary with frequency, then the selectivity can be expressed simply in terms of either the wavelength or the frequency since

$$\frac{f}{f_0} - \frac{f_0}{f} = \frac{\lambda_0}{\lambda} - \frac{\lambda}{\lambda_0}. \tag{34}$$

However, when the velocity in the transmission line varies with frequency, equation 34 does not hold true, and the expression relating the two parameters is more complicated. In the case of the rectangular waveguide

$$\lambda_g = \frac{c}{\sqrt{f^2 - f_{cw}^2}} \tag{35}$$

where c is the velocity of light in vacuum, f_{cw} is the cutoff frequency of the waveguide, $f_{cw} = \frac{c}{2a}$ and a is the width of the waveguide.

It can be shown readily that the frequency parameter can he expressed in terms of the wavelength, thus

$$\left(\frac{f}{f_0} - \frac{f_0}{f}\right) = \left(\frac{\lambda_{g0}}{\lambda_g} - \frac{\lambda_g}{\lambda_{g0}}\right)\left(\frac{\lambda_a}{\lambda_g}\right)\left(\frac{\lambda_{a0}}{\lambda_{g0}}\right) \tag{36}$$

where λ_a is the wavelength in free space.

For narrow percentage bands, this reduces to the approximate relation

$$\left(\frac{f}{f_0} - \frac{f_0}{f}\right) \doteq \left(\frac{\lambda_{g0}}{\lambda_g} - \frac{\lambda_g}{\lambda_{g0}}\right)\left(\frac{\lambda_{a0}}{\lambda_{g0}}\right)^2. \tag{37}$$

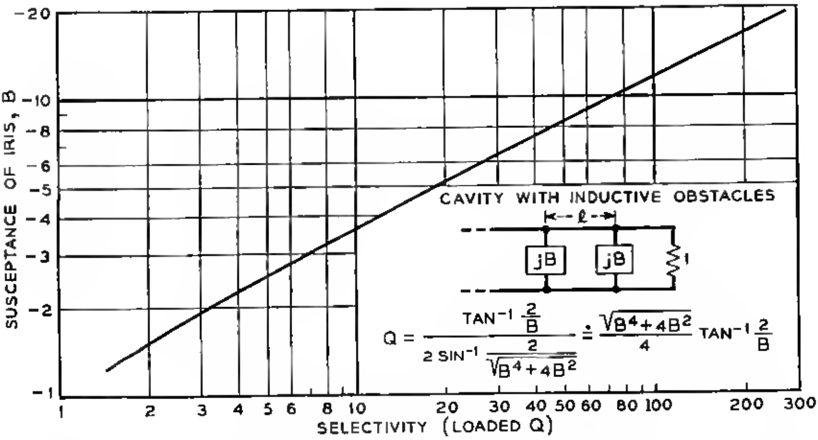


Fig. 11—The relation between loaded Q and normalized susceptance. (Inductive obstacles)

This states, in effect, that the percentage bandwidth is greater in terms of wavelength than in terms of frequency, by the square of the ratio of the wavelengths in the guide and in free space. The selectivity in terms of the frequencies and wavelength ratio thus becomes

$$Q \doteq \frac{1}{\left(\frac{f_{c2}}{f_0} - \frac{f_0}{f_{c1}}\right)} \cdot \left(\frac{\lambda_{a0}}{\lambda_{g0}}\right)^2 = \frac{f_0}{(f_{c2} - f_{c1})} \cdot \left(\frac{\lambda_{a0}}{\lambda_{g0}}\right)^2 \tag{38}$$

This is the selectivity that is plotted as a function of B in Figures 11 and 12.

Excess Phase and Connecting Lines

The excess phase of this type of cavity is taken into account by adding the lengths of line, ℓ' (see Fig. 9), which have a length given by the relation (See Appendix I)

$$\tan\frac{4\pi\ell'}{\lambda_{g0}} = -\left(\frac{2}{B}\right). \tag{39}$$

Combining Eq. 31 and Eq. 39 and solving for ℓ' in terms of ℓ

$$\ell' = \frac{\lambda_{g0}}{4} - \frac{\ell}{2} \tag{40}$$

where ℓ is the length of the cavity and λ_{g0} is the resonant wavelength in the line.

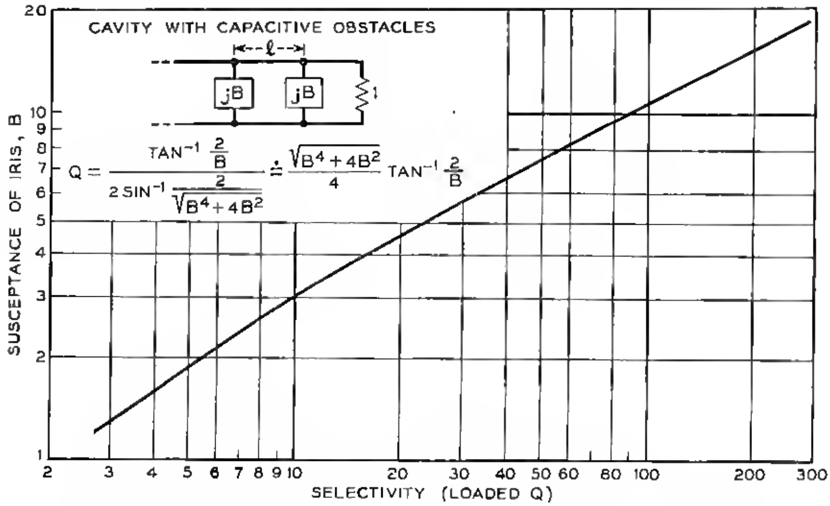


Fig. 12—The relation between loaded Q and normalized susceptance. (Capacitive obstacles)

Thus, when this length, corresponding to the excess phase of the cavity resonator, is absorbed in the length of line connecting two cavities together, the correct total connecting length becomes

$$\ell_c = (2m + 1)\frac{\lambda_{g0}}{4} - \ell_1' - \ell_2'$$

$$= \frac{\ell_1 + \ell_2}{2} - \frac{\lambda_g}{4} + m\frac{\lambda_g}{2} \tag{41}$$

where ℓ_1 and ℓ_2 are the lengths of the cavities and m is any integer including zero.

Obstacles in Waveguides

The three properties of the cavity—the resonant frequency, the selectivity and the excess phase—are given in Equations 31, 32 and 39, regardless of the sign of the normalized susceptance, B. In the case where the obstacles are inductive, B is negative; and where the obstacles are capacitive, B is positive.

Further explanation is needed to distinguish between these two important cases. First consider the case where inductive obstacles are used. $\tan \frac{2\pi \ell}{\lambda_{g0}}$ is negative and the cavity length lies between a quarter and a half wavelength (plus any multiple of half wavelength). The selectivity, as given by equation (32), is plotted on Fig. 11 for the fundamental mode. The excess phase is positive, and the added lengths, ℓ', of Fig. 9 are positive. The connecting lines between two such cavities are then slightly less than a quarter wavelength (or odd multiple thereof).

Next consider the case where the obstacles are capacitive. $\tan \frac{2\pi \ell}{\lambda_{g0}}$ is positive and the cavity length lies between zero and a quarter wavelength (plus any multiple of half wavelengths). The selectivity as given by equation (32) is plotted in Fig. 12 for cavity lengths lying between a half wavelength and three quarters wavelength. The excess phase is negative and the added lengths, ℓ_1' of Fig. 9 are negative. The connecting lines between two such cavities are then slightly longer than a quarter wavelength (or odd multiple thereof).

Susceptance of Obstacles

The Equations (31), (32) and (39) give the resonant wavelength, the selectivity (in terms of wavelength) and the excess phase as functions of the normalized susceptance of the obstacles which form the ends of the cavity, and a knowledge of this susceptance as a function of the geometrical configuration of the obstacle is necessary to complete the design of the filter. At low frequencies, conventional coils and condensers can be used to form the discontinuities in the transmission line; while at high frequencies, transmission line stubs can be used.[14] In the microwave region, where waveguides are employed, obstacles having the shapes shown in Figures 13, 14, and 15 can be used.[15]

Inductive Vanes

Figure 13 shows a plane metallic obstacle, transversely located across a rectangular waveguide, with a centrally located rectangular opening extending completely across the waveguide in a direction parallel to the electric vector. For thin obstacles, the normalized susceptance can be calculated from the approximate formula,[15]

$$B \doteq -\frac{\lambda_g}{a} \cot^2 \frac{\pi d}{2a} \tag{42}$$

where λ_g is the wavelength in the waveguide, a is the width of the waveguide, and d is the width of the iris opening.

When the iris is constructed of material of finite thickness, τ, the expression for the susceptance is more complicated,[10, 11] and the equivalent circuit becomes a four-terminal network with both shunt and series elements. The equivalent shunt susceptance of this network can be obtained experimentally by measuring the insertion loss of the iris, from which a curve such as shown in Fig. 16 can be computed. These data* were taken for irises .050″ thick in waveguide having internal dimensions of 0.872″ × 1.872″ in the frequency range around 4000 mc. The ordinate is a parameter, K, from which the normalized susceptance is calculated:

$$B = K\left(\frac{\lambda_g}{2a}\right). \tag{43}$$

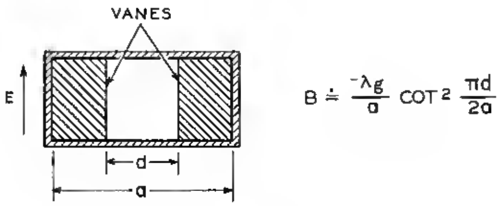


Fig. 13—One type of inductive obstacle in rectangular waveguide.

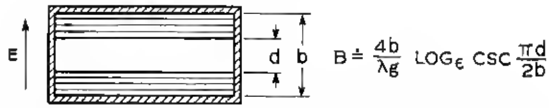


Fig. 14—One type of capacitive obstacle in rectangular waveguide.

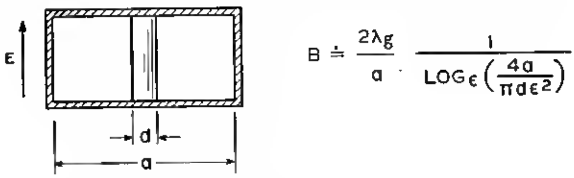


Fig. 15—Another type of inductive obstacle consists of a cylindrical post.

Along the abscissa is plotted the ratio of iris opening to width of the waveguide.

It can be demonstrated tbat for values of K from -1 to -20, the equivalent iris opening is approximately the actual opening less the thickness of the metal sheet. For practical purposes, when the susceptance lies between -1.5 and -30, it is often sufficient to use the approximation,

$$B \doteq -\frac{\lambda_g}{a} \cot^2\left(\frac{\pi(d - \tau)}{2a}\right) \tag{44}$$

where τ is the thickness of the iris.

* Data supplied by Mr. L. C. Tillotson of Bell Telephone Laboratories.

Capacitive Irises

The normalized susceptance of infinitely thin capacitive obstacles, as illustrated in Fig. 14, may be calculated by the approximate relation[15]

$$B \doteq \frac{4b}{\lambda_g} \log_\epsilon \operatorname{cosec} \frac{\pi d}{2b} \tag{45}$$

where b is the height of the waveguide, λ_g is the wavelength in the waveguide, and d is the width of the iris opening.

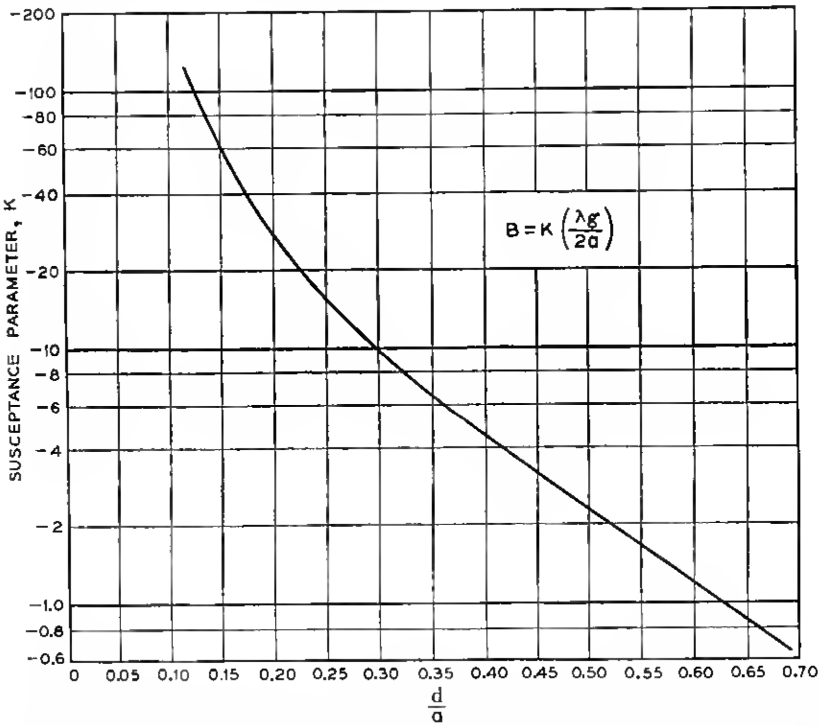


Fig. 16—Experimentally determined curve of normalized susceptance of inductive irises.

As with the inductive vanes, the normalized susceptance is a function of the iris thickness and may be calculated from the approximate formula[15]

$$B \doteq B_0 + \frac{2\pi\tau}{\lambda_g}\left(\frac{b}{d} - \frac{d}{b}\right) \tag{46}$$

where B_0 is the normalized susceptance of the infinitely thin iris, and τ is the iris thickness.

For best results, the irises should be designed from experimentally determined curves, however.

Inductive Posts

The normalized susceptance of the round cylindrical inductive post, centrally located in the waveguide parallel to the electric vector, may be calculated from the approximate formula[11, 15, 16]

$$B = -\frac{2\lambda_g}{a}\frac{1}{\log_\epsilon\left(\frac{4a}{\pi d\epsilon^2}\right)} \tag{47}$$

where a is the width of the guide, and d is the post diameter.

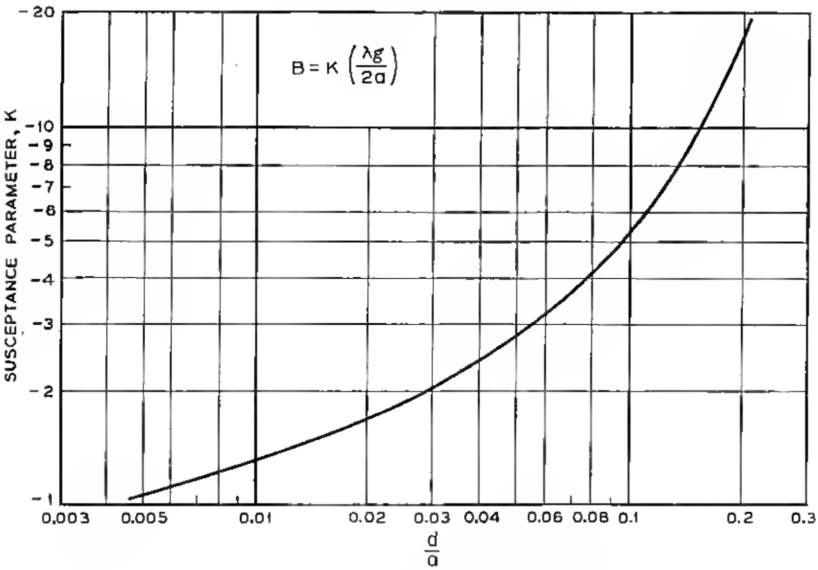


Fig. 17—Experimentally determined curve of normalized susceptance of inductive posts

The experimentally determined values of susceptance are somewhat less than the values calculated by the formula (47). The difference is less than 20% when $\frac{d}{a}$ is less than 0.08. A curve of experimentally determined values is plotted in Fig. 17, the data being taken in rectangular waveguide 0.872″ × 1.872″ at a frequency near 4000 mc.*

The normalized susceptance of posts is also a function of their position in the waveguide, the susceptance decreasing as the posts are moved off center. This feature may be used when it is desired to make all the posts in a filter

* Data supplied by Mr. A. E. Bowen of Bell Telephone Laboratories.

from stock of a given diameter. The expression for the normalized susceptance of off-center posts is given by the relation[11, 16]

$$B \doteq -\frac{2\lambda_g}{a}\,\frac{1}{\sec^2\dfrac{\pi s}{a}\left[\log_\epsilon\left(\dfrac{4a}{\pi d\epsilon^2}\cos\dfrac{\pi s}{a}\right)\right]} \tag{48}$$

where s is the distance off center.

Experimental Data

The principles of waveguide filter design as outlined in the foregoing have been used in several applications. For example the channel branching filters in the New York-Boston microwave radio relay link consist of two resonant cavities separated by the equivalent of $\frac{3}{4}$ wavelength sections of waveguide. The transmitting modulators in this relay system also use two-chamber filters to separate the wanted sideband from the unwanted sideband. The transmission band in each of these applications was 10 mc

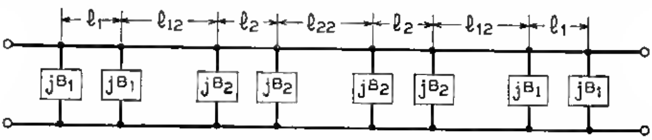


Fig. 18—Diagram of a transmission line filter consisting of four resonant cavities and three connecting lengths of line.

and the image frequency or the unwanted sideband which was to be reflected was 130 mc away.

In another case the requirements were that the standing wave ratio should be less than 0.64 db over a band of 20 mc and more than 28 db 30 mc on each side of the midband frequency. The design formulae indicated that a filter consisting of four cavities would be needed. These, then, would take the general configuration shown in Fig. 18, where the first and last cavities are formed by the obstacles jB_1 and the length of line ℓ_1, while the two middle cavities are formed by the obstacles jB_2 and the length ℓ_2. The lengths ℓ_{12} and ℓ_{22} correspond to the transforming sections of transmission line which connect the cavities together. The loaded Q's required to meet the specifications turned out to be $Q_1 = 12.25$ and $Q_2 = 30.0$, after allowance had been made for the selectivities of the $\frac{3}{4}$ wavelength connecting sections. Assuming that the cavities would be formed with inductive obstacles, as shown schematically in Fig. 19, the susceptances to obtain these selectivities were obtained from Fig. 11 based on equation 32. This gave

$$B_1 = -4.08$$

$$B_2 = -6.36$$

These susceptances were realized with centrally located round posts, for which the data of Fig. 17 has been plotted, and this filter was constructed according to the calculated dimensions which are shown in Fig. 20. Each of the four cavities was tuned separately to resonance near midband by adjusting a capacitive plug located in the center of each. The characteristic then obtained is plotted in Fig. 21, which shows that the standing wave

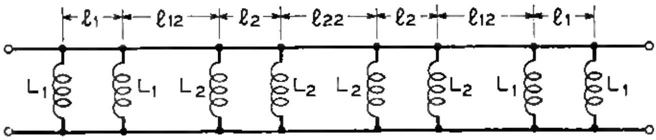


Fig. 19—A four-cavity filter which utilizes inductive obstacles.

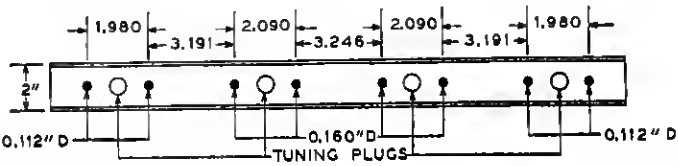


Fig. 20—The calculated dimensions for a four-cavity maximally-flat filter in 0.872" x 1.872" rectangular waveguide.

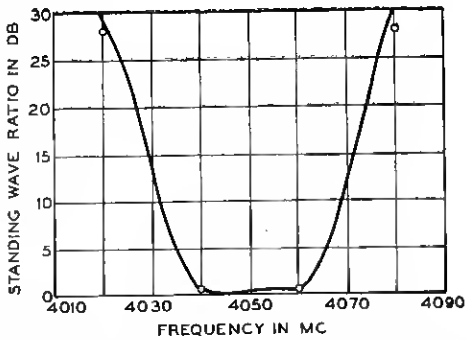


Fig. 21—Measured characteristic of four-cavity filter of Figure 20.

ratio met the design points quite well. These are shown as circles in the figure. The insertion loss of this filter was less than 0.7 db over a 25-mc band and less than 0.3-db at midband.

Another maximally-flat waveguide filter consisting of fifteen resonant cavities gave an insertion loss of two decibels at midband, 4-db loss at 20-mc bandwidth and 40-db loss at 30-mc bandwidth. The input standing wave ratio was less than 1.0 db over a 20-mc band. Its characteristics are plotted in Figs. 22 and 23. This excellent performance is remarkable in

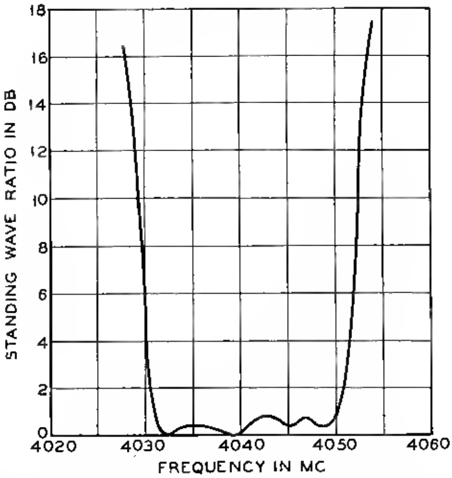


Fig. 22—Measured standing wave ratio of maximally-flat filter consisting of fifteen resonant cavities.

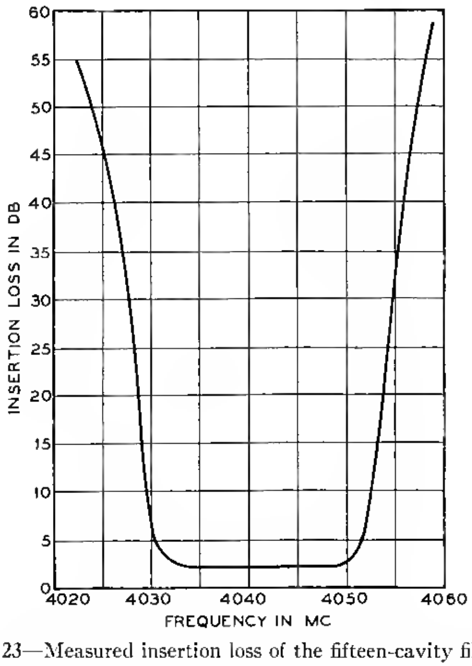


Fig. 23—Measured insertion loss of the fifteen-cavity filter.

view of the difficulties that might be encountered in constructing and aligning a filter consisting of 75 discontinuities and 29 lengths of waveguide. Its physical length (over 80″) may be seen in the photograph of Fig. 24.

Fig. 24—The fifteen cavity filter.

The theoretical treatment of maximally-flat filters presented here has ignored the dissipation in the elements. Better agreement between expected and observed characteristics would he obtained if this had heen taken into account. The ohservation of .3-dh loss and 2-db loss in the four-cavity and the fifteen-cavity maximally-flat filters is indicative of the amounts of added insertion loss to be expected because of dissipation in the elements. In addition to the increased loss at midhand, we should expect a rounding of the insertion loss characteristic near the cutoff frequencies, and a broadening of the standing wave characteristic at frequencies well beyond cutoff. In many applications, however, these effects can be ignored.

Concluding Remarks

In the foregoing, the design of maximally-flat hand-pass filters has heen treated in detail. The treatment of other types of band-pass and hand-rejection filters is beyond the scope of the present paper, although much of the material presented here may be of use in designing such filters. In fact, almost any filter consisting of a ladder network of inductive and capacitive elements in series and in shunt can he simulated in waveguides hy following these principles. Emphasis on the maximally-flat filter has been deliberate for two reasons: it gives a type of transmission characteristic that is useful in microwave work; it is simple to design.

Acknowledgement

Many members of the Holmdel Radio Research Laboratories have influenced the evolution of the technique of huilding waveguide filters. A firm foundation was laid hy the pioneering work of G. C. Southworth[17] and A.G. Fox. Intimate association with W. D. Lewis and L. C. Tillotson fostered many stimulating discussions which clarified douhtful issues. The comments of M. D. Brill and S. Darlington are deeply appreciated. The untiring effort of R. H. Brandt in making skillfully the accurate measurements which were necessary is gratefully acknowledged as is also the invaluable support of all the many other people without whose cooperation and confidence the work would have been impossihle.

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APPENDIX I

A cavity resonator, consisting of a length of transmission line, ℓ, at each end of which there is an unvarying susceptance, jB, is approximately equivalent to a tuned circuit, consisting of an inductance, L, and a capacity, C, in parallel located at the center of a short length of transmission line, $2\ell'$, when these two conditions are satisfied:

(1) The square root of L over C is equal to the surge impedance of the transmission line divided by twice the loaded Q of the cavity.

(2) The sum of the lengths of the two transmission lines ℓ and $2\ell'$ is equal to a half wavelength at resonance.

The first of these conditions follows from equation 3 of the text above, and the proof of the second condition will be given in the following analysis, based on the schematic drawing of Figures 9 and 10. In this analysis, the loaded Q of the cavity is derived in terms of the susceptance of the obstacles at its ends.

Since the cavity and the tuned circuit are both symmetrical it is adequate to consider but one half of each in establishing the equivalence. Then by setting the short circuit admittance of one equal to the other and setting the open circuit admittance of one equal to the other, the necessary relationships are derived.

The following symbols will be used in addition to those used in the text:

Y_{SC} = Normalized admittance, short circuited.

Y_{OC} = Normalized admittance, open circuited.

The subscripts 1 and x refer to the cavity and the equivalent tuned circuit respectively.

$$\theta_1 = \frac{2\pi}{\lambda_g} \cdot \frac{\ell}{2}$$

$$\theta_x = \frac{2\pi}{\lambda_g} \cdot \ell'$$

The short-circuited admittances of half the cavity and half the tuned circuit are

$$Y_{SC1} = j(B_1 - \cot \theta_1) \tag{A1}$$

$$Y_{SCx} = -j \cot \theta_x \tag{A2}$$

while the open-circuited admittances are

$$Y_{OC1} = j(B_1 + \tan \theta_1) \tag{A3}$$

$$Y_{OCx} = \frac{j\left(\dfrac{B_x}{2} + \tan \theta_x\right)}{1 - \dfrac{B_x}{2} \tan \theta_x}. \tag{A4}$$

Putting $Y_{SC1} = Y_{SC2}$

$$\tan \theta_x = \frac{1}{\cot \theta_1 - B_1}. \tag{A5}$$

Putting A5 in A4 and setting $Y_{OC1} = Y_{OCx}$

$$B_1 + \tan \theta_1 = \frac{\dfrac{B_x}{2} + \dfrac{1}{\cot \theta_1 - B_1}}{1 - \dfrac{B_x}{2} \dfrac{1}{\cot \theta_1 - B_1}}. \tag{A6}$$

Solving for B_x we have

$$B_x = -B_1(B_1 \sin 2\theta_1 - 2 \cos 2\theta_1) \tag{A7}$$

which becomes

$$B_x = \sqrt{B_1^4 + 4B_1^2} \sin \frac{2\pi \ell}{\lambda_{g0}} \left(\frac{\lambda_{g0}}{\lambda_g} - 1\right) \tag{A8}$$

where

$$\frac{2\pi \ell}{\lambda_{g0}} = \text{arc tan } \frac{2}{B_1}. \tag{A9}$$

Equation A9 gives the requirements for resonance.

The expression for the loaded Q is

$$Q = \frac{\dfrac{2\pi \ell}{\lambda_{g0}}}{\dfrac{2\pi \ell}{\lambda_{gc2}} - \dfrac{2\pi \ell}{\lambda_{gc1}}}. \tag{A10}$$

The cutoff wavelengths are obtained when B_x is equal to $\pm$ 2 and we bave from equation A8

$$\frac{2\pi\ell}{\lambda_{gc1}} = \frac{2\pi\ell}{\lambda_{g0}} - \text{arc sin}\,\frac{2}{B_1\sqrt{B_1^2 + 4}}, \tag{A11}$$

$$\frac{2\pi\ell}{\lambda_{gc2}} = \frac{2\pi\ell}{\lambda_{g0}} + \text{arc sin}\,\frac{2}{B_1\sqrt{B_1^2 + 4}}, \tag{A12}$$

from which we obtain

$$Q = \frac{\text{arc tan}\,\frac{2}{B}}{2\,\text{arc sin}\,\frac{2}{\sqrt{B^4 + 4B^2}}} \doteq \frac{\sqrt{B_1^4 + 4B_1^2}}{4}\,\text{arc tan}\,\frac{2}{B}. \tag{A13}$$

This gives the loaded Q of the cavity in terms of the susceptance of the end obstacles.

To derive the length corresponding to the excess phase of the cavity, let the short-circuited admittances be equal by equating equations A1 and A2, and let the wavelength be the resonant wavelength of the cavity, and we bave

$$B_1 - \cot\theta_{10} = -\cot\theta_{xo}. \tag{A14}$$

From equation A9

$$B_1 = 2\cot 2\theta_{10} \tag{A15}$$

so tbat

$$2\cot 2\theta_{10} - \cot\theta_{10} = -\cot\theta_{x0}. \tag{A16}$$

But

$$2\cot 2\theta_{10} - \cot\theta_{10} = -\tan\theta_{10} \tag{A17}$$

bence

$$\tan\theta_{10} = \cot\theta_{x0} \tag{A18}$$

or

$$\theta_{10} + \theta_{x0} = \frac{\pi}{2} \tag{A19}$$

That is

$$\frac{2\pi}{\lambda_{g0}} \cdot \frac{\ell}{2} + \frac{2\pi\ell'}{\lambda_{g0}} = \frac{\pi}{2}$$

whence

$$\ell + 2\ell' = \frac{\lambda_{g0}}{2} \tag{A20}$$

which proves the second condition mentioned above, namely, that the sum of the lengths of the transmission lines in the cavity and its equivalent circuit is equal to a half wavelength.

The normalized admittance of the cavity terminated in the surge admittance of the guide can be written in terms of its loaded Q and a wavelength variable as

$$YR \doteq 1 + j2Q\left[2\left(\frac{\lambda_{g0}}{\lambda_g}\right) - 1\right]. \tag{A21}$$

This expression is obtained from equations A8 and A13 by making the assumption that the bandwidth is narrow so that the sine of the angle in equation A8 can be replaced by the angle. This admittance is referred to a point slightly inside the cavity, i.e. a distance ℓ' inside.

The similarity between this expression and the corresponding one for the parallel resonant circuit consisting of lumped elements is evident. (See eq. 7 of the text.)

$$YR = 1 + j2Q\left[\frac{f}{f_o} - \frac{f_0}{f}\right] \tag{A22}$$

In the case of the cavity the bracketed term is a wavelength variable; in the case of the tuned circuit it is a frequency variable.

The loss function for maximally-flat filters in waveguides becomes

$$\frac{P_0}{P_L} \doteq 1 + \left[Q_T\, 2\left(\frac{\lambda_{g0}}{\lambda_g} - 1\right)\right]^{2n}. \tag{A23}$$

The loaded Q's of the cavities taper sinusoidally from one end of the filter to the other so that

$$Q_r = Q_T \sin\left(\frac{2r - 1}{2n}\right)\pi. \tag{A24}$$